AP® CALCULUS BC
2007 SCORING GUIDELINES

Question 1

Let R be the region in the first and second quadrants bounded above by the graph of $y = \dfrac{20}{1+x^2}$ and below by the horizontal line $y = 2$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the x-axis.

(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles. Find the volume of this solid.

$\dfrac{20}{1+x^2} = 2$ when $x = \pm 3$	1 : correct limits in an integral in (a), (b), or (c)
(a) Area $= \displaystyle\int_{-3}^{3} \left(\frac{20}{1+x^2} - 2 \right) dx = 37.961$ or 37.962	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(b) Volume $= \pi \displaystyle\int_{-3}^{3} \left(\left(\frac{20}{1+x^2} \right)^2 - 2^2 \right) dx = 1871.190$	$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(c) Volume $= \dfrac{\pi}{2} \displaystyle\int_{-3}^{3} \left(\frac{1}{2} \left(\frac{20}{1+x^2} - 2 \right) \right)^2 dx$ $= \dfrac{\pi}{8} \displaystyle\int_{-3}^{3} \left(\frac{20}{1+x^2} - 2 \right)^2 dx = 174.268$	$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

1A1

CALCULUS BC

SECTION II, Part A

Time—45 minutes

Number of problems—3

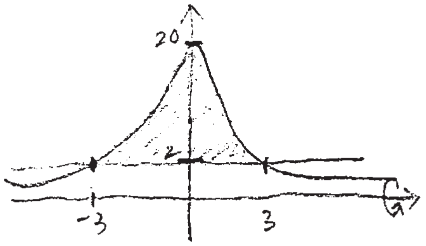
A graphing calculator is required for some problems or parts of problems.

Work for problem 1(a)

$$\text{Area} = \int_{-3}^{3}\left(\frac{20}{1+x^2}-2\right)dx$$

/

$$= 37.9618$$



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Continue problem 1 on page

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Work for problem 1(b)

$R = \frac{20}{1+x^2}$ $r = 2$

$V = \pi \int_{-3}^{3} \left[\left(\frac{20}{1+x^2} \right)^2 - 4 \right] dx$

$= 1871.1901$

Work for problem 1(c)

$D = \frac{20}{1+x^2} - 2$

$r = \frac{\frac{20}{1+x^2} - 2}{2} = \frac{10}{1+x^2} - 1$

$V = \int_{-3}^{3} \frac{\pi \left(\frac{10}{1+x^2} - 1 \right)^2}{2} dx = \frac{\pi}{2} \int_{-3}^{3} \left(\frac{10}{1+x^2} - 1 \right)^2 dx$

$= 174.2685$

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1B

CALCULUS BC

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.

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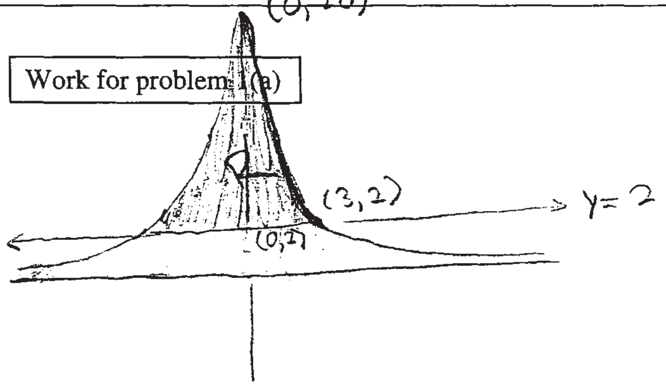
Work for problem 1(a)



$$\cancel{\begin{aligned} y &= \frac{20}{1+x^2} \\ 1+x^2 &= \frac{20}{y} \end{aligned}}$$

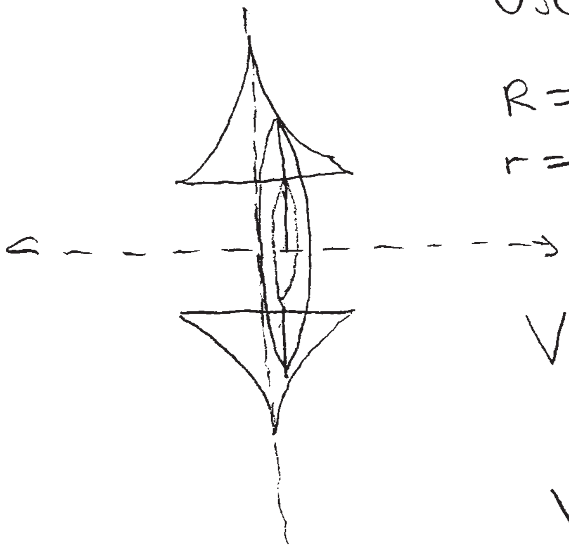
$$A_R = 2\int_0^3 \left[\left(\frac{20}{1+x^2}\right) - 2\right] dx$$

$$A_R = 37.962$$

Continue problem 1 on page

1B$_2$

Work for problem 1(b)



Use washers method.

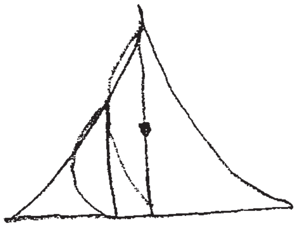
$$R = \frac{20}{1+x^2}$$

$$r = 2$$

$$V = 2\left[\pi \int_0^3 \left(\left(\frac{20}{1+x^2}\right)^2 - 2^2\right) dx\right]$$

$$V = 1871.190$$

Work for problem 1(c)



$$r = \frac{\left(\frac{20}{1+x^2}\right)}{2} = \frac{10}{1+x^2}$$

$$V = 2\left[\frac{1}{2}\pi \int_0^3 \left(\frac{10}{1+x^2}\right)^2\right]$$

$$V = 243.324$$

GO ON TO THE NEXT PAGE.

1C1

CALCULUS AB

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.

Work for problem 1(a)

R shaded region bounded by $y = \frac{20}{1+x^2}$ & $y = 2$

Intersect at $(-3,2)$ & $(3,2)$

A of R =

$$\int_{-3}^{3} \left(\frac{20}{1+x^2} - 2 \right) dx$$

Area of shaded region R = $\boxed{37.962 \text{ units}^2}$

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Continue problem 1 on page 5.

$1C_2$

Work for problem 1(b)

[redacted]

$$V = \pi \int_a^b \left(R(x) - r(x)\right)^2 dx$$

$$V = \pi \int_{-3}^{3} \left(\frac{20}{1+x^2} - 2\right)^2 dx$$

$$\boxed{V = 1394.148 \text{ units}^3}$$

Work for problem 1(c)

A of semicircle $\frac{1}{2}\pi r^2$

$$V: \int_{-3}^{3} \frac{1}{2}\pi r^2 \quad \text{where } r = \frac{20}{1+x^2}$$

$$V = \int_{-3}^{3} \frac{1}{2}\pi \left(\frac{20}{1+x^2}\right)^2$$

$$\boxed{V : 973.294 \text{ unit}^3}$$

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AP® CALCULUS BC
2007 SCORING COMMENTARY

Question 1

Overview

This problem presented students with a region bounded above by the graph of a function and below by a horizontal line. Because no picture was provided, students were expected to graph the function on their calculators or use their knowledge of rational functions to sketch the graph, and then identify the appropriate region from their graph. The points of intersection of the graph and the horizontal line could be found either algebraically or with the calculator. Students needed to find, in part (a), the area of the region; in part (b), the volume of the solid generated when the region was rotated about the x-axis; and in part (c), the volume of the solid above the region for which the cross sections perpendicular to the x-axis were semicircles.

Sample: 1A
Score: 9

The student earned all 9 points.

Sample: 1B
Score: 6

The student earned 6 points: the region point, 2 points in part (a), 3 points in part (b), and no points in part (c). In part (a) the student sets up the definite integral using symmetry and earned the region point. The student has the correct integrand, which earned the first point in part (a). The answer is correct to three decimal places and earned the second point. In part (b) the student continues to use symmetry in defining the limits of integration. The correct integrand earned the first 2 points. The answer is correct to three decimal places and earned the third point. In part (c) the student did not earn any points because the radius is incorrect.

Sample: 1C
Score: 3

The student earned 3 points: the region point, 2 points in part (a), no points in part (b), and no points in part (c). In part (a) the student earned the region point by using the correct limits of integration. The student has the correct integrand, which earned the first point in part (a). The answer is correct to three decimal places and earned the second point. In part (b) the student did not earn any points because the washer method is not used correctly. In part (c) the student did not earn any points because the radius is incorrect.

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2007 SCORING GUIDELINES

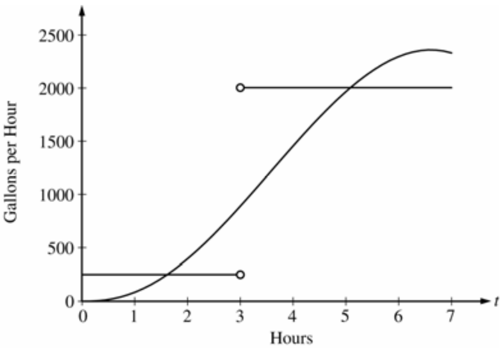
Question 2

The amount of water in a storage tank, in gallons, is modeled by a continuous function on the time interval $0 \le t \le 7$, where t is measured in hours. In this model, rates are given as follows:

(i) The rate at which water enters the tank is $f(t) = 100t^2 \sin\left(\sqrt{t}\right)$ gallons per hour for $0 \le t \le 7$.

(ii) The rate at which water leaves the tank is

$$g(t) = \begin{cases} 250 & \text{for } 0 \le t < 3 \\ 2000 & \text{for } 3 < t \le 7 \end{cases} \text{ gallons per hour.}$$



The graphs of f and g, which intersect at $t = 1.617$ and $t = 5.076$, are shown in the figure above. At time $t = 0$, the amount of water in the tank is 5000 gallons.

(a) How many gallons of water enter the tank during the time interval $0 \le t \le 7$? Round your answer to the nearest gallon.

(b) For $0 \le t \le 7$, find the time intervals during which the amount of water in the tank is decreasing. Give a reason for each answer.

(c) For $0 \le t \le 7$, at what time t is the amount of water in the tank greatest? To the nearest gallon, compute the amount of water at this time. Justify your answer.

(a) $\int_0^7 f(t)\,dt \approx 8264$ gallons

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

(b) The amount of water in the tank is decreasing on the intervals $0 \le t \le 1.617$ and $3 \le t \le 5.076$ because $f(t) < g(t)$ for $0 \le t < 1.617$ and $3 < t < 5.076$.

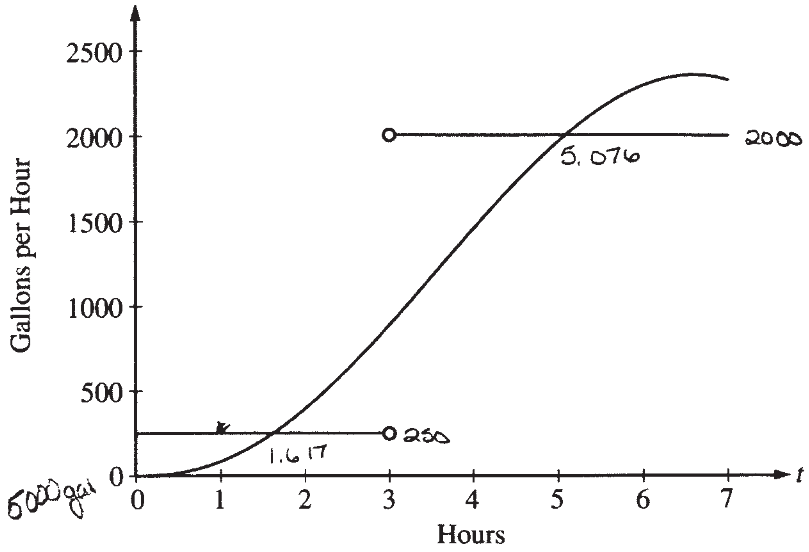
$$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$$

(c) Since $f(t) - g(t)$ changes sign from positive to negative only at $t = 3$, the candidates for the absolute maximum are at $t = 0,\ 3,$ and 7.

t (hours)	gallons of water
0	5000
3	$5000 + \int_0^3 f(t)\,dt - 250(3) = 5126.591$
7	$5126.591 + \int_3^7 f(t)\,dt - 2000(4) = 4513.807$

The amount of water in the tank is greatest at 3 hours. At that time, the amount of water in the tank, rounded to the nearest gallon, is 5127 gallons.

$$5 : \begin{cases} 1 : \text{identifies } t = 3 \text{ as a candidate} \\ 1 : \text{integrand} \\ 1 : \text{amount of water at } t = 3 \\ 1 : \text{amount of water at } t = 7 \\ 1 : \text{conclusion} \end{cases}$$



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Work for problem 2(a)

$\int_0^7 f(t)dt$

8264 gallons

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Continue problem 2 on page 7.

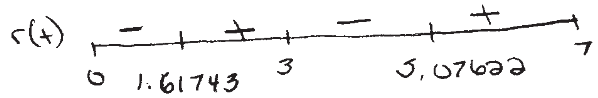
Work for problem 2(b)

$r(t) = f(t) - g(t)$

$r(t) = f(t) - 250$ for $0 \le t < 3$ $\quad$ $r(t) = f(t) - 2000$ for $3 < t \le 7$

$0 = f(t) - 250$ $\quad$ $0 = f(t) - 2000$

$t = 1.61743$ $\quad$ $t = 5.07622$



The amount of water in the tank is decreasing on (0, 1.61743) b/c r(t) that is defined for $0 \le t < 3$ is negative. It is also decreasing on (3, 5.07622) b/c r(t) that is defined for $3 < t \le 7$ is negative.

Work for problem 2(c)

Relative max at time t=3 b/c r(t) changes from (+) to (−)

$5000 + \int_0^3 r(t)\,dt =$ $\quad$ (0, 5000)

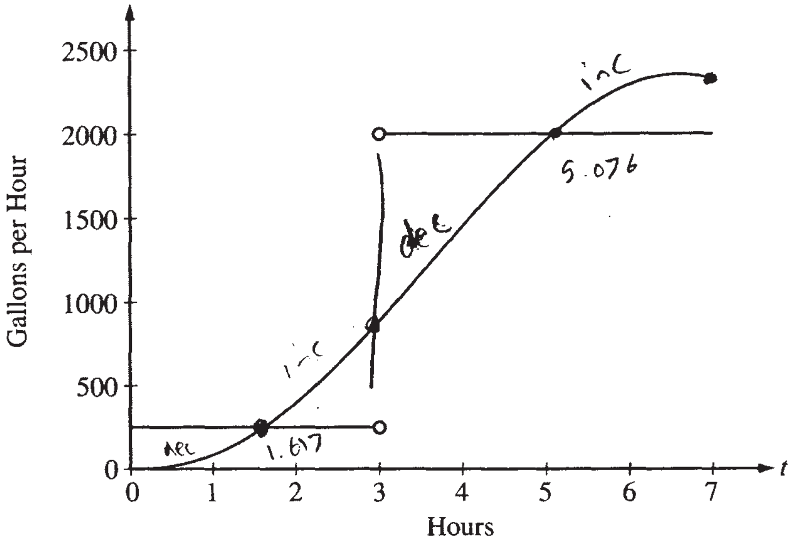
5126.591 $\quad$ (3, 5126.591)

$\quad$ (7, 4513.807)

$5000 + \int_0^3 r(t)\,dt + \int_3^7 r(t)\,dt =$

4513.807

The amount of water in the tank is greatest at time t=3 which is a relative max for the function of the amount of water in the tank. At the boundaries of the interval, the amount of water in the tank is equal to 5000, at t=0, and 4514, at t=7. At t=3, the amount of water in the tank is 5127 gallons, which is greater than the values at the boundaries.



t=0, 5000 gals

Work for problem 2(a)

a) $\int_0^7 100t^2 \sin(\sqrt{t})\,dt = \boxed{8264 \text{ gallons}}$

Continue problem 2 on page 7.

Work for problem 2(b)

b) water is decreasing (0, 1.617), because $g(t) > f(t)$ on that interval.

water is decreasing (3, 5.076), because $g(t) > f(t)$ on that interval

Work for problem 2(c)

$t=0$, 5000 gals

c) $\underline{h(t)} = f(t) - g(t)$

$h(7) = f(7) - g(7)$

$100(7)^2 \sin\sqrt{7} - 2000$

$5000 + \int_0^7 100t^2 \sin\sqrt{t} - 2000 \, dt = -736.19$

$5000 + \int_0^0 100t^2 \sin\sqrt{t} - 2000 \, dt = 5000$

$5000 + \int_0^3 100t^2 \sin\sqrt{t} - 2000 \, dt = -123.409$

$h(0) = 5000$

$h(7) = -736.19$

$h(3) = -123.409$

At $t=0$, the amount of water is greatest

At $t=0$, the amount of water is 5000 gallons

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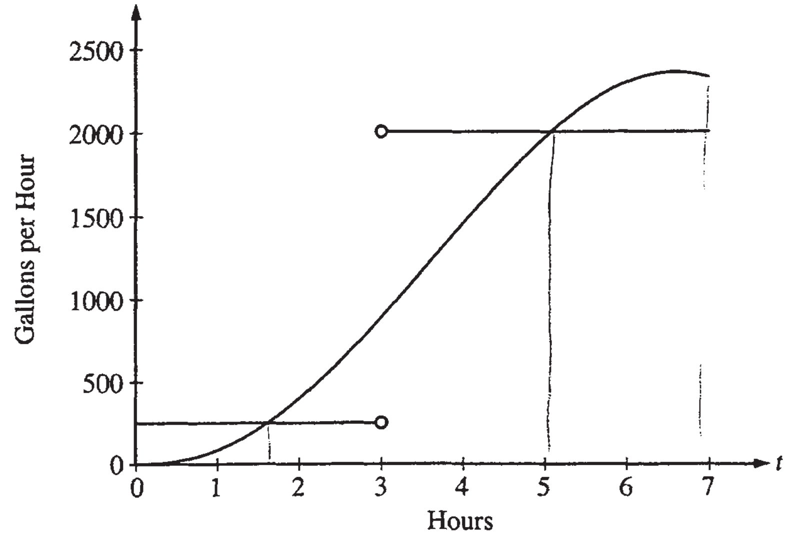

Work for problem 2(a)

$\int_0^7 100t^2\sin(\sqrt{t})\,dt \approx 8264$ gallons

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Continue problem 2 on page 7.

2C$_2$

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Work for problem 2(b)

The amount in the tank is decreasing on the intervals (0, 1.617) ∪ (1.617, 5.076) because on those intervals the rate of water entering is less than the rate of the water leaving.

Work for problem 2(c)

At time t=7. because the water entering is greater than the water leaving is at a max for the interval.

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AP® CALCULUS BC
2007 SCORING COMMENTARY

Question 2

Overview

This problem presented students with two functions that modeled the rates, in gallons per hour, at which water entered and left a storage tank. The latter function was piecewise-constant. Graphs of each function were also provided. In part (a) students had to use a definite integral to find the total amount of water that entered the tank over a given time interval. Part (b) measured their abilities to compare the two rates to find, with justification, the time intervals during which the amount of water in the tank was decreasing. This could be determined directly from the graphs and the information given about the points of intersection, but students needed to be able to handle the point of discontinuity in the piecewise-defined function. Part (c) asked for the time at which the amount of water was at an absolute maximum and the value of this maximum amount to the nearest gallon. Again, dealing with the critical point at the discontinuity was an important part of the analysis, as was using the net rate of change during the first three hours and during the last four hours to compute the total amount of water in the tank at $t = 3$ and $t = 7,$ respectively.

Sample: 2A
Score: 9

The student earned all 9 points. Note that in part (b), in the presence of the correct numerical values in reported intervals, errors in the use of open, closed, or half-open interval notation were ignored, and the student earned the interval point. The student defines $r(t)$ in part (b) and that definition may be used in part (c).

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), and 2 points in part (c). In part (a) the student gives the correct integral and the correct answer and earned both the integral and answer points. In part (b) the student gives the correct intervals and a correct reason and earned both the interval and reason points. In part (c) the student considers $t = 0,$ $t = 3,$ and $t = 7$ as candidates for the absolute maximum with the evaluation of the three integrals presented. The student therefore considers $t = 3$ a candidate and earned the first point. The student presents the integrand $f(t) - g(t)$ in an integral and earned the second point. The incorrect use of the rule for $g(t)$ results in incorrect values for the amount of water at time $t = 3$ and $t = 7,$ so the third and fourth points in part (c) were not earned; and the student presents an incorrect conclusion, so the fifth point was not earned.

Sample: 2C
Score: 3

The student earned 3 points: 2 points in part (a), 1 point in part (b), and no points in part (c). In part (a) the student gives the correct integral and the correct answer and earned both the integral and answer points. In part (b) the student reports an incorrect left endpoint on the second interval, so the interval point was not earned. The student provides correct reasoning and earned the reason point. In part (c) the student never considers $t = 3$ as a candidate for the absolute maximum, so the first point was not earned. No integrand in an integral is presented, and the amounts of water at $t = 3$ and $t = 7$ are not calculated. The second, third, and fourth points in part (c) were not earned, and the student presents an incorrect conclusion, so the fifth point was not earned.

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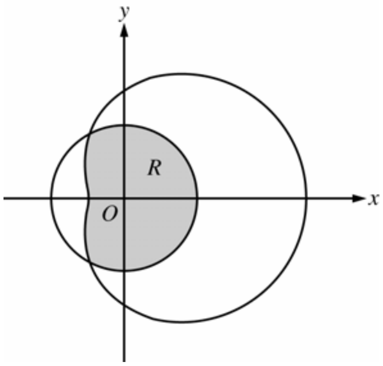
Question 3



The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

(a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

(b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(a) Area $= \frac{2}{3}\pi(2)^2 + \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (3 + 2\cos\theta)^2 \, d\theta$

$= 10.370$

$$4 : \begin{cases} 1 : \text{area of circular sector} \\ 2 : \text{integral for section of limaçon} \\ \quad 1 : \text{integrand} \\ \quad 1 : \text{limits and constant} \\ 1 : \text{answer} \end{cases}$$

(b) $\left.\frac{dr}{dt}\right|_{\theta=\pi/3} = \left.\frac{dr}{d\theta}\right|_{\theta=\pi/3} = -1.732$

The particle is moving closer to the origin, since $\frac{dr}{dt} < 0$ and $r > 0$ when $\theta = \frac{\pi}{3}$.

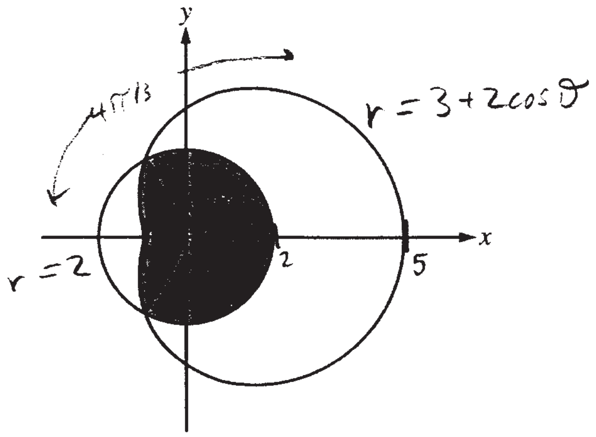
$$2 : \begin{cases} 1 : \left.\frac{dr}{dt}\right|_{\theta=\pi/3} \\ 1 : \text{interpretation} \end{cases}$$

(c) $y = r\sin\theta = (3 + 2\cos\theta)\sin\theta$

$\left.\frac{dy}{dt}\right|_{\theta=\pi/3} = \left.\frac{dy}{d\theta}\right|_{\theta=\pi/3} = 0.5$

The particle is moving away from the x-axis, since $\frac{dy}{dt} > 0$ and $y > 0$ when $\theta = \frac{\pi}{3}$.

$$3 : \begin{cases} 1 : \text{expression for } y \text{ in terms of } \theta \\ 1 : \left.\frac{dy}{dt}\right|_{\theta=\pi/3} \\ 1 : \text{interpretation} \end{cases}$$



Work for problem 3(a)

$$\text{Area} = \frac{1}{2}\int_{-2\pi/3}^{2\pi/3} (2)^2\, d\theta + \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (3+2\cos\theta)^2\, d\theta$$

$$\approx \boxed{10.370}$$

Continue problem 3 on page 9.

3A$_2$

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Work for problem 3(b)

$t = \theta = \pi/3$

$$\frac{dr}{dt} = \frac{d}{dt}\left[3+2\cos t\right] = -2\sin t$$

$$\left.\frac{dr}{dt}\right|_{t=\pi/3} = -2\sin\frac{\pi}{3} = \boxed{-\sqrt{3}}$$

At time $t=\frac{\pi}{3}$, The particle is approaching the origin at a rate of $\sqrt{3}$.

Work for problem 3(c)

$t = \theta = \pi/3$

Since $r = 3+2\cos\theta$,

$$y = r\sin\theta = (3+2\cos\theta)(\sin\theta)$$

$$y(t) = (3+2\cos t)(\sin t)$$

$$\frac{dy}{dt} = 4\cos^2 t + 3\cos t - 2$$

$$\left.\frac{dy}{dt}\right|_{t=\frac{\pi}{3}} = 4\cos^2\left(\frac{\pi}{3}\right) + 3\cos\left(\frac{\pi}{3}\right) - 2 = \boxed{\frac{1}{2}}$$

At time $t=\pi/3$, the particle's height is increasing at a rate of 1/2. Thus, the particle is moving Up at a rate of 1/2 (but it might also be moving horizontally).

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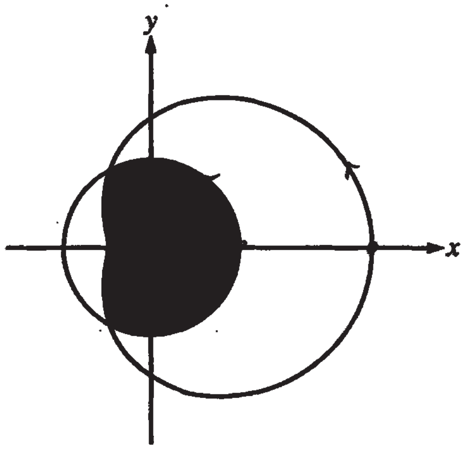
END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

3B1



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Work for problem 3(a)

Let $r_1 = 2$ $r_2 = 3+2\cos\theta$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$$A = \frac{1}{2}\int_0^{2\pi/3}(r_1)^2 d\theta + \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(r_2)^2 d\theta + \frac{1}{2}\int_{4\pi/3}^{2\pi}(r_1)^2 d\theta =$$

$$= \frac{1}{2}\left(8.37758 + 3.985787 + 8.37758\right)$$

$$= 10.37047$$

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Continue problem 3 on page 9.

Work for problem 3(b)

$(x(t), y(t))$ $\qquad \theta = 0$ when $t = 0$ $\qquad$ $r = x\cos\theta$
$r = y\sin\theta$

$$\frac{dr}{dt} = \frac{dr}{d\theta}, \quad r = 3 + 2\cos\theta$$

$$\frac{dr}{dt} = \frac{dr}{d\theta} = -2\sin\theta$$

$$\left.\frac{dr}{dt}\right|_{\theta = \frac{\pi}{3}} = \left.\frac{dr}{d\theta}\right|_{\theta = \frac{\pi}{3}} = -2\sin\frac{\pi}{3} = -\sqrt{3}$$

Particle is moving down and to the right @ $\theta = \frac{\pi}{3}$

$$\frac{dr}{d\theta} = -\frac{dx}{d\theta}\sin\theta$$
$$-\sqrt{3} = -\frac{dx}{d\theta}\left(\frac{\sqrt{3}}{2}\right)$$
$$\frac{dr}{d\theta} = \frac{dy}{d\theta}\cos\theta$$
$$-\sqrt{3} = \frac{1}{2}\frac{dy}{d\theta}$$
$$\frac{dx}{d\theta} = 2$$
$$\frac{dy}{d\theta} = -2\sqrt{3}$$

$r^2 = x^2 + y^2$

Work for problem 3(c)

$$\frac{dy}{dt} = \frac{dy}{d\theta}$$

$$y = r\sin\theta$$
$$y = (3 + 2\cos\theta)\sin\theta$$
$$\frac{dy}{d\theta} = -2\sin\theta\sin\theta + \cos\theta(3 + 2\cos\theta)$$
$$\frac{dy}{d\theta} = -2\sin^2\theta + 3\cos\theta + 2\cos^2\theta$$
$$\left.\frac{dy}{d\theta}\right|_{\theta = \frac{\pi}{3}} = -2\left(\frac{\sqrt{3}}{2}\right)^2 + 3\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^2$$
$$= \frac{-3}{2} + \frac{3}{2} + \frac{1}{4} = \boxed{\frac{1}{4}}$$

particle

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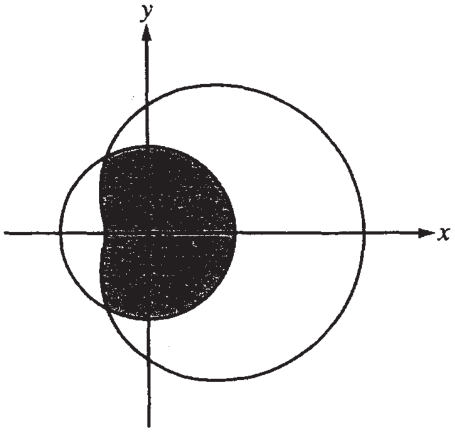
END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

3C1



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Work for problem 3(a)

$$\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (2)^2 - (3+2\cos\theta)^2 \, d\theta = 2.196$$

Continue problem 3 on page

$3C_2$

Work for problem 3(b)

$y = r\sin\theta$

$x = r\cos\theta$

$\frac{dy}{d\theta} = \frac{d}{d\theta}\left[(3+2\cos\theta)\sin\theta\right]$

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Do not write beyond this border.

Work for problem 3(c)

$\frac{dy}{dt}\Big|_{\theta=\pi/3} = r\cos\theta + \sin\theta \cdot \frac{dr}{d\theta} \quad = (3+2\cos\theta)\cos\theta + \sin\theta(-2\sin\theta)$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Question 3

Overview

This problem presented students with two curves described in polar coordinates with r as a function of θ. The values of θ for which the two curves intersect were also given. Part (a) judged students' ability to find the area of a region bounded by curves described in polar coordinates. Parts (b) and (c) involved the behavior of a particle moving with nonzero velocity along one of the polar curves (and with constant angular velocity $\frac{d\theta}{dt} = 1$, although students did not need to know that to answer the questions). Students were asked to compute $\frac{dr}{dt}$ and $\frac{dy}{dt}$ at a specific value of θ, and then to interpret their answers in terms of the motion of this particle. The interesting aspect of the motion was that at this instant, the distance of the particle from the origin was decreasing while its vertical distance from the x-axis was increasing.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: 4 points in part (a), 1 point in part (b), and 1 point in part (c). The student earned all 4 points in part (a). The student correctly evaluates the derivative at $\theta = \frac{\pi}{3}$ and earned the first point in part (b), but the student does not interpret this value as the particle moving closer to the origin so the second point was not earned. In part (c) the student correctly states $y = r\sin\theta$ and earned the first point. The student incorrectly evaluates the derivative in part (c) and then fails to give an interpretation of the value of the derivative.

Sample: 3C
Score: 3

The student earned 3 points: 2 points in part (a), no points in part (b), and 1 point in part (c). The student did not earn the first point for the circular area, but the integrand and the limits and constant points were earned for a section of the limaçon relevant to the problem. The student did not earn the answer point. In part (b) the student does not evaluate the derivative and so was not eligible for the interpretation point. In part (c) the student states the correct form of the derivative of $y = r\sin\theta$ and earned the first point. The student does not evaluate the derivative at $\theta = \frac{\pi}{3}$ and so did not earn the second point. The student was not eligible for the answer point.

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Question 4

Let f be the function defined for $x > 0$, with $f(e) = 2$ and f', the first derivative of f, given by $f'(x) = x^2 \ln x$.

(a) Write an equation for the line tangent to the graph of f at the point $(e, 2)$.

(b) Is the graph of f concave up or concave down on the interval $1 < x < 3$? Give a reason for your answer.

(c) Use antidifferentiation to find $f(x)$.

(a) $f'(e) = e^2$

An equation for the line tangent to the graph of f at the point $(e, 2)$ is $y - 2 = e^2(x - e)$.

$$2 : \begin{cases} 1 : f'(e) \\ 1 : \text{equation of tangent line} \end{cases}$$

(b) $f''(x) = x + 2x \ln x.$

For $1 < x < 3$, $x > 0$ and $\ln x > 0$, so $f''(x) > 0$. Thus, the graph of f is concave up on $(1, 3)$.

$$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$$

(c) Since $f(x) = \int \left(x^2 \ln x\right) dx$, we consider integration by parts.

$$u = \ln x \qquad dv = x^2\, dx$$
$$du = \frac{1}{x}\, dx \qquad v = \int \left(x^2\right) dx = \frac{1}{3}x^3$$

Therefore,

$$\begin{aligned} f(x) &= \int \left(x^2 \ln x\right) dx \\ &= \frac{1}{3}x^3 \ln x - \int \left(\frac{1}{3}x^3 \cdot \frac{1}{x}\right) dx \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C. \end{aligned}$$

Since $f(e) = 2$, $2 = \frac{e^3}{3} - \frac{e^3}{9} + C$ and $C = 2 - \frac{2}{9}e^3$.

Thus, $f(x) = \frac{x^3}{3} \ln x - \frac{1}{9}x^3 + 2 - \frac{2}{9}e^3$.

$$4 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{uses } f(e) = 2 \\ 1 : \text{answer} \end{cases}$$

4A1

NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Work for problem 4(a)

$f'(x) = x^2 \ln x$

$f'(e) = e^2 \ln(e) = e^2$

$\boxed{y-2 = e^2(x-e)}$

Do not write beyond this border.

Work for problem 4(b)

$f'(x) = x^2 \ln x$

$f''(x) = (x^2)(\frac{1}{x}) + (2x)\ln x = x + 2x\ln x$

$f''(1) = 1 + 2(1)\ln(1) = 1 + 0 = 1$

$f''(2) = 2 + 2(2)\ln(2) = 2 + 4\ln(2)$

$f''(3) = 3 + 2(3)\ln(3) = 3 + 6\ln(3)$

f is concave up on $1 < x < 3$ because f" is positive on this interval

Continue problem 4 on page 11

NO CALCULATOR ALLOWED

Work for problem 4(c)

$f'(x) = x^2 \ln x$

let $u = \ln x$, $dv = x^2\,dx$

$du = \frac{1}{x}\,dx$ $v = \frac{1}{3}x^3$

$f(x) = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2\,dx$

$f(x) = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

$f(e) = 2$

$2 = \frac{1}{3}(e)^3 \ln(e) - \frac{1}{9}(e)^3 + C$

$2 = \frac{e^3}{3} - \frac{e^3}{9} + C$

$\frac{3e^3}{9} - \frac{e^3}{9} = \frac{2e^3}{9}$

$2 = \frac{2e^3}{9} + C$

$f(x) = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + \left(\frac{18 - 2e^3}{9}\right)$

$2 - \frac{2e^3}{9} = C$

$\frac{18}{9} - \frac{2e^3}{9} = C$

$C = \frac{18 - 2e^3}{9}$

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GO ON TO THE NEXT PAG

NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Do not write beyond this border.

Work for problem 4(a)

$f'(x) = x^2 \ln x$

$f'(e) = e^2$

$y - 2 = e^2(x - e)$

$y = e^2x - e^3 + 2$

Work for problem 4(b)

$f''(x) = x^2\left(\frac{1}{x}\right) + \ln x(2x)$

$= x + 2x\ln x = 0$

$f''(x) = x(1+2\ln x)$

$f(x)$	1	3
$f''(x)$		

$x(1+2\ln x) = 0$

$x = 0,\ \ln x = -1/2$

$x = e^{-1/2} = \frac{1}{e^{1/2}}$

Continue problem 4 on page 1

4B2

NO CALCULATOR ALLOWED

Work for problem 4(c)

$f'(x) = x^2 \ln x$

$u = \ln x \qquad dv = x^2 dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^3}{3}$

$\frac{x^3}{3} \ln x - \int \frac{x^2}{3} dx$

$\frac{x^3}{3} \ln x - \frac{x^3}{9}$

$f(x) = \frac{x^3}{3} \left(\ln x - \frac{1}{3} \right)$

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GO ON TO THE NEXT PAG

4C1

NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

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Work for problem 4(a)

$m = f'(e) = e^2$

$e = 2e^2 + b$

$b = e - 2e^2$

$\boxed{y = e^2 x + (e - 2e^2)}$

Work for problem 4(b)

$f''(x) = \ln(x) \cdot 2x + x$

$\ln(2) \cdot 4 + 2 \Rightarrow +$

concave up, b/c $f'(2)$ is positive

Continue problem 4 on page 1

4C$_2$

Work for problem 4(c)

$$f(x) = \int x^2 \ln(x)\,dx$$

Tabular Integration

$$\begin{tikzcd}
x^2 \arrow[dr, bend left, "+"] & \ln x \\
2x \arrow[dr, "-"] & 1/x \\
2 \arrow[dr, "+"] & -2/x^2 \\
0 & +2/x^3
\end{tikzcd}$$

$$f(x) = \frac{x^2}{x} + \frac{2x \cdot 2}{x^2} + \frac{2 \cdot 3}{x^3} + C$$

$$\boxed{f(x) = x + \frac{4}{x} + \frac{6}{x^3} + C}$$

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GO ON TO THE NEXT PAGI

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Question 4

Overview

This problem presented students with the derivative of a function and an initial value. In part (a) students had to use the given information to write an equation for the tangent line at the initial value. In part (b) they needed to determine whether the graph of the function was concave up or concave down on a given interval and to justify their answer. Part (c) asked students to use antidifferentiation to find an explicit formula for the function. This involved using integration by parts, a BC-only topic.

Sample: 4A
Score: 9

The student earned all 9 points. Note that in part (b) the student includes the necessary reference to the interval $1 < x < 3$ when discussing the concavity of the graph of f.

Sample: 4B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), and 2 points in part (c). In part (a) the student has a correct numerical value for the slope and uses it to determine an equation of the tangent line. In part (b) $f''(x) = x + 2x\ln x$ is correct, and the first 2 points were earned. The third point was not earned because there is no comment about the concavity of the graph of f. In part (c) the student correctly uses integration by parts and earned 2 points. Since there is no substitution for $f(e) = 2$ or a final solution for $f(x)$, the last 2 points were not earned. The missing $+C$ does not affect the antiderivative points but was required for the remaining points.

Sample: 4C
Score: 3

The student earned 3 points: 1 point in part (a) and 2 points in part (b). In part (a) the student earned the point for a correct $f'(e)$. The student's equation of the tangent line is incorrect. In part (b) $f''(x) = x + 2x\ln x$ is correct, and the first 2 points were earned. The explanation is not sufficient since the student states that $f'(2)$ is positive instead of $f''(2)$ is positive. For this problem, students were allowed to evaluate f' at only two points to determine whether f' is increasing or decreasing on the interval $1 < x < 3$. In part (c) the student's use of integration by parts is incorrect.

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2007 SCORING GUIDELINES

Question 5

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

(a) $r(5.4) \approx r(5) + r'(5)\Delta t = 30 + 2(0.4) = 30.8$ ft Since the graph of r is concave down on the interval $5 < t < 5.4$, this estimate is greater than $r(5.4)$.	$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{conclusion with reason} \end{cases}$
(b) $\frac{dV}{dt} = 3\left(\frac{4}{3}\right)\pi r^2 \frac{dr}{dt}$ $\left.\frac{dV}{dt}\right\rvert_{t=5} = 4\pi(30)^2 2 = 7200\pi \text{ ft}^3/\text{min}$	$3 : \begin{cases} 2 : \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$
(c) $\int_0^{12} r'(t)\,dt \approx 2(4.0) + 3(2.0) + 2(1.2) + 4(0.6) + 1(0.5)$ $= 19.3$ ft $\int_0^{12} r'(t)\,dt$ is the change in the radius, in feet, from $t = 0$ to $t = 12$ minutes.	$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{explanation} \end{cases}$
(d) Since r is concave down, r' is decreasing on $0 < t < 12$. Therefore, this approximation, 19.3 ft, is less than $\int_0^{12} r'(t)\,dt$.	1 : conclusion with reason
Units of ft^3/min in part (b) and ft in part (c)	1 : units in (b) and (c)

NO CALCULATOR ALLOWED

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

Work for problem 5(a)

$r(5)=30$

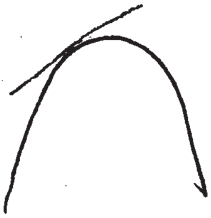
$y-30=2\cdot(x-5)$

$y-30=2\cdot(5.4-5)$

$y-30=.8$

$\boxed{y=30.8\text{ ft}}$

this estimate is greater than the actual Value because the graph of r(t) is concave down



Work for problem 5(b)

$V=\frac{4}{3}\pi r^3$

$\frac{dV}{dt}=\frac{4}{\cancel{3}}\cdot\cancel{3}\pi r^2\frac{dr}{dt}$

$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$

$\frac{dV}{dt}=4\pi\cdot30^2\cdot2$

$\frac{dV}{dt}=4\pi\cdot900\cdot2$

$\frac{dV}{dt}=7200\pi\ \text{ft}^3/\text{minute}$

Continue problem 5 on page 13.

5A 2

NO CALCULATOR ALLOWED

Work for problem 5(c)

$$\int_0^{12} r'(t)\,dt = 2\cdot 4 + 3\cdot 2 + 2\cdot 1.2 + 4\cdot .6 + 1\cdot .5$$

$$= 8+6+2.4+2.4+.5$$

$$= 8+6+4.8+.5$$

$$= 14+5.3$$

$$= \boxed{19.3 \text{ feet}}$$

this is the change in the radius of the baloon from $t = 0$ min to $t = 12$ min

Work for problem 5(d)

the estimation is less than the actual value because $r'(t)$ is decreasing on the interval $0 < t < 12$

NO CALCULATOR ALLOWED

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

Work for problem 5(a)

$L_n = f(a) + f'(a)(x-a)$

$= 30 + 2(x-5) = 30+2x-10 = 20+2x$

Work for problem 5(b)

$V = \frac{4}{3}\pi r^3$

$\frac{dv}{dt} = \frac{\cancel{3}}{\cancel{3}} 4\pi r^2 \frac{dr}{dt}$

$\frac{dv}{dt} = 4\pi(30)^2(2) = 4\pi(900)(2) = \boxed{7200\pi \frac{\text{feet}^3}{\text{minute}}}$

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Continue problem 5 on page 13.

$5B_2$

NO CALCULATOR ALLOWED

Work for problem 5(c)

$$RRAM = 1(.5) + \overset{4}{\cancel{8}}(.6) + 2(1.2) + 3(2) + 4(2) =$$

$$= .5 + 2.4 + 2.4 + 6 + 8 = \boxed{19.3 \text{ feet}}$$

$\int_0^{12} r'(t)\,dt$ is the sum of the area under the curve of $r'(t)$. It shows the radius of the balloon in feet ~~at~~ 12 minutes after it ~~go~~ begins to expand.

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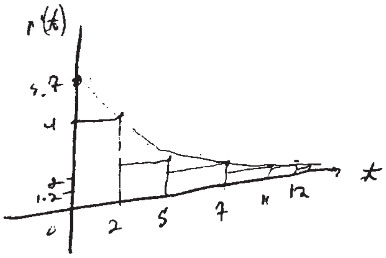
Work for problem 5(d)



The approximation in part c is less than $\int_0^{12} r'(t)\,dt$ because the ~~areas of a~~ approximated areas within each subinterval fall below the graph of $r'(t)$ because $r'(t)$ is a decreasing function.

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

Work for problem 5(a)

$R = 30 + 2 * (t-5)$

$R = 30 + 2\cdot 5(t-5)$

$R = 30 + 10(t-5)$

$R = 10t - 20$

$R = 10(5.4) - 20$

$R = 34$ feet

This is less than the true value because $r'(t)$ is (+), so the radius is increasing. If r is measured at $t=5$, the measurement will be less than r when $r = 5.4$

Work for problem 5(b)

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 dr$

$\frac{dV}{dt} = 4\pi(30)^2(2)$

$\frac{dV}{dt} = 7200\pi \text{ ft}^3/\text{min}$

Continue problem 5 on page 13.

5C2

NO CALCULATOR ALLOWED

Work for problem 5(c)

$S = 2(4+2+1.2+0.6+0.5)$

$S = 16.6$ feet

$\int_0^{12} r'(t)dt = r(t) =$ the radius of the balloon at time t.

Work for problem 5(d)

part (c) approximation is greater than $\int_0^{12} r'(t)dt$. Since the radius is increasing, the right value of each approximation is the highest value for each interval, so the approximation is greater.

GO ON TO THE NEXT PAGE.

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Question 5

Overview

The problem presented students with a table of values for the rate of change of the radius of an expanding spherical balloon over a time interval of 12 minutes. Students were told that the radius was modeled by a twice-differentiable function whose graph was concave down. Part (a) asked students to use a tangent line approximation to estimate the radius of the balloon at a specific time and to determine if the estimate was greater than or less than the true value. This tested their ability to use the information about the concavity of the graph of the radius to make the appropriate conclusion about the behavior of the tangent line. In part (b) students had to handle the related rate of change of the volume, given information about the rate of change of the radius. In part (c) students had to recognize the definite integral as the total change, in feet, of the radius of the balloon from time $t = 0$ minutes to time $t = 12$ minutes and approximate the value of this integral using a right Riemann sum and the data in the table. Part (d) asked students to decide if this approximation was greater than or less than the true value of the definite integral. Again, they were required to use the information about the concavity of the graph of the radius to make the appropriate conclusion about the behavior of the graph of the derivative. Units of measure were important in parts (b) and (c).

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 3 points in part (b), 1 point in part (c), 1 point in part (d), and the units point. In part (a) the student finds the tangent line approximation correctly but does not compute the estimate nor state a conclusion or reason. Thus no points were earned. In part (b) the student correctly finds $\frac{dV}{dt}$ using the chain rule and is therefore eligible for the answer point. This answer is also correct. In part (c) the student finds the correct approximation using a right Riemann sum but fails to provide a correct explanation—this integral represents the *change* in radius, not the radius, after 12 minutes. In part (d) the student correctly identifies the reason that the approximation is less than the actual value: $r'(t)$ is decreasing. The student earned the units point.

Sample: 5C
Score: 4

The student earned 4 points: 3 points in part (b) and the units point. In part (a) the student calculates the estimate for $r(5.4)$ incorrectly and states that $r'(t)$ is positive rather than decreasing. In part (b) the student earned all 3 points for correct use of the chain rule and the correct calculation of the result. In part (c) the student makes an error in calculating the right Riemann sum and does not refer to the *change* in radius. In part (d) the student states that the approximation is greater than the definite integral rather than less. The student earned the units point.

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Question 6

Let f be the function given by $f(x) = e^{-x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use your answer to part (a) to find $\lim\limits_{x \to 0} \dfrac{1 - x^2 - f(x)}{x^4}$.

(c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}\,dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2}\,dt$.

(d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}\,dt$ by less than $\dfrac{1}{200}$.

(a) $$e^{-x^2} = 1 + \frac{\left(-x^2\right)}{1!} + \frac{\left(-x^2\right)^2}{2!} + \frac{\left(-x^2\right)^3}{3!} + \cdots + \frac{\left(-x^2\right)^n}{n!} + \cdots$$
$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots$$

$$3 : \begin{cases} 1 : \text{two of } 1, -x^2, \dfrac{x^4}{2}, -\dfrac{x^6}{6} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$$

(b) $$\frac{1 - x^2 - f(x)}{x^4} = -\frac{1}{2} + \frac{x^2}{6} + \sum_{n=4}^{\infty} \frac{(-1)^{n+1} x^{2n-4}}{n!}$$

Thus, $\lim\limits_{x \to 0} \left(\dfrac{1 - x^2 - f(x)}{x^4} \right) = -\dfrac{1}{2}$.

1 : answer

(c) $$\int_0^x e^{-t^2}\,dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots \right) dt$$
$$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$$

Using the first two terms of this series, we estimate that $\int_0^{1/2} e^{-t^2}\,dt \approx \dfrac{1}{2} - \left(\dfrac{1}{3}\right)\left(\dfrac{1}{8}\right) = \dfrac{11}{24}$.

$$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{estimate} \end{cases}$$

(d) $\left| \int_0^{1/2} e^{-t^2}\,dt - \dfrac{11}{24} \right| < \left(\dfrac{1}{2}\right)^5 \cdot \dfrac{1}{10} = \dfrac{1}{320} < \dfrac{1}{200}$, since

$\int_0^{1/2} e^{-t^2}\,dt = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{n!(2n+1)}$, which is an alternating series with individual terms that decrease in absolute value to 0.

$$2 : \begin{cases} 1 : \text{uses the third term as the error bound} \\ 1 : \text{explanation} \end{cases}$$

NO CALCULATOR ALLOWED

6A1

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✓ **Work for problem 6(a)**

$$e^{x} = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

$$e^{-x^2} = 1+\frac{-x^2}{1!}+\frac{(-x^2)^2}{2!}+\frac{(-x^2)^3}{3!}+\dots = 1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\dots+\frac{(-1)^n x^{2n}}{n!}+\dots$$

✓ **Work for problem 6(b)**

$$\lim_{x\to 0}\frac{\cancel{1}-\cancel{x^2}-\cancel{1}+\cancel{x^2}-\frac{x^4}{2!}+\frac{x^6}{3!}+\dots}{x^4} = \lim_{x\to 0} -\frac{1}{2}+\frac{\cancel{x^2}}{3!}+\dots = \boxed{-\frac{1}{2}}$$

Do not write beyond this border.

Continue problem 6 on page 15.

6A2

Work for problem 6(c)

$$\int_0^x e^{-t^2}\,dt = 0 + x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!} + \ldots$$

$$\int_0^{1/2} e^{-t^2}\,dt \approx 1/2 - 1/24 = \boxed{\frac{11}{24}}$$

$$\lim_{n\to\infty}\left|\frac{x^2\cancel{x^{2n+1}}}{(2n+3)(n+1)\cancel{n!}}\cdot\frac{\cancel{(2n+1)}\cancel{n!}}{\cancel{x^{2n+1}}}\right| = 0 < 1$$

$$\uparrow$$
$$\infty$$

it is converges for all x (including 1/2)

Work for problem 6(d)

the taylor series for $\int_0^{1/2} e^{-t^2}\,dt$ is an alternating series

① nth term goes to 0 $\quad \lim_{n\to\infty} \frac{(1/2)^{2n+1}}{(2n+1)\,n!} = 0$ ✓

② $|a_n| > |a_{n+1}| \quad |a_n| = \frac{1/2^{(2n+1)}}{(2n+1)\,n!} > \frac{1/2^{2n+1}\cdot 1/4}{(2n+3)(n+1)!} = |a_{n+1}|$ ✓

numerator decreased

denominator increased

$$|E| < |a_{n+1}| = \frac{(1/2)^5}{5\cdot 2!} = \frac{1}{320} < \frac{1}{200}$$

$$\therefore\ |E| < 1/200$$

NO CALCULATOR ALLOWED

6B

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Work for problem 6(a)

$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ replace x with $-x^2$

$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{n!}$ Gen term $= \frac{(-1)^n x^{2n}}{n!}$

1st ~~four~~ non zero terms $1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!}$

Work for problem 6(b)

$\lim_{x\to 0} f(0) = 1 \lim_{x \to 0} \frac{1 - x^2 - 1}{x^4} = \frac{0}{0} \overset{\textcircled{L}}{=} \frac{-2x}{4x^3} \overset{\textcircled{L}}{=} \frac{-2 -2}{12x^2}$

$\lim_{x\to 0} \frac{-4}{12x^2} = -\infty$

Continue problem 6 on page 15

6B2

Work for problem 6(c)

$\int f(x) \approx x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^6}{36}$

$\int_0^{1/2} f(x) \approx \frac{1}{2} - \frac{(\frac{1}{2})^3}{3} - 0 - \frac{(0)^3}{3}$

$\approx \frac{1}{2} - \frac{(\frac{1}{2})^3}{3}$

$\approx \frac{1}{2} - \frac{\frac{1}{8}}{3}$

$\approx \frac{1}{2} - \frac{1}{24}$

$\approx 11/24$

Work for problem 6(d)

The taylor series for $\int f(x)$ is alternating. the error is less than the n+1 term because Term n+1 < Term n and $\lim_{n \to \infty}$ of terms = 0 and as I said It's alternating

So Error ≤ n+1 term

$|\text{error}| \leq 3^{rd}$ term

$\leq \frac{(\frac{1}{2})^5}{10}$

$|E_1| \leq \frac{\frac{1}{32}}{10}$

Error $\leq \frac{1}{320} \rightarrow < \frac{1}{200}$

Since error is less than $\frac{1}{320}$, which is less than 1/200 our aproximation wi'll not be off by more than 1/200.

6C1

NO CALCULATOR ALLOWED

Work for problem 6(a)

$$\frac{f^n(c)(x-c)^n}{n!}$$

		x=0		
~~0~~	~~e^{-x^2}~~	~~1~~	~~x^0~~	~~0!~~
~~1~~	~~$-2xe^{-x^2}$~~	~~1~~	~~x^1~~	~~1!~~
~~2~~	~~$4x^2e^{-x^2}$~~	~~1~~	~~x^2~~	~~2!~~
~~3~~	~~$-8x^3e^{-x^2}$~~	~~1~~	~~x^3~~	~~3!~~
~~4~~	~~$16x^4e^{-x^2}$~~	~~1~~	~~x^4~~	~~4!~~

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{-x^2} = 1 - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \frac{x^5}{4!}$$

$$f(x) = 1 - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!}$$

Work for problem 6(b)

$$\cancel{x}=\cancel{x^2}\ \cancel{-x}\ \cancel{+x^2} - \frac{x^3}{2!} + \frac{x^4}{3!}$$

4

$$\lim_{x\to 0} \frac{1 - x^2 - \left(1 - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!}\right)}{x^4}$$

$$\lim_{x\to 0} \frac{-\frac{x^3}{2} + \frac{x^4}{6}}{x^4} = \lim_{x\to 0} -\frac{1}{2}\cdot\frac{\cancel{x^3}}{\cancel{x^4}} + \frac{1}{6}\cdot\frac{x^4}{x^4}$$

$$\lim_{x\to 0} -\frac{1}{2x} + \frac{1}{6} = \text{DNE}$$

Continue problem 6 on page 15.

6C2

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Work for problem 6(c)

$$e^{-x^2} = 1 - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!}$$

$$\int_0^x e^{-t^2} = x - \frac{x^3}{3} + \frac{x^4}{4\cdot 2!} - \frac{x^5}{5\cdot 3!}$$

$$\int_0^{1/2} e^{-t^2}\,dt = \left[x - \frac{x^3}{3}\right]_0^{1/2} = \frac{1}{2} - \frac{(1/2)^3}{3} = \frac{1}{2} - \frac{\frac{1}{8}}{\frac{3}{1}}$$

$$\frac{1\cdot 12}{2\cdot 12}\ \frac{1}{24} = \boxed{\frac{11}{24}}$$

Work for problem 6(d)

The next term in the series is used to find the error in a taylor polynomial. For this one it is $\frac{x^4}{4\cdot 2!}$. By pluggin in (0.1) for x, the error is a value less than 1/200.

$$\frac{(0.1)^4}{8} < 1/200$$

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2007 SCORING COMMENTARY

Question 6

Overview

This problem dealt with Taylor series. Part (a) assessed students' abilities to find the first four nonzero terms and the general term of the Taylor series for $f(x) = e^{-x^2}$. Although it would be possible to do this by computing derivatives of the function *f*, it was expected that students would start with the known Taylor series for the exponential function and use substitution. Part (b) asked for a limit of an indeterminate form $\left(\frac{0}{0}\right)$ involving the function *f*. Students were asked to use their answer about the Taylor series for *f* rather than using repeated applications of L'Hospital's Rule. Part (c) required students to formally manipulate the Taylor series for *f* in a way that could be used to estimate the value of a definite integral. Part (d) asked students to explain why the value of the estimate differed from the actual value of the definite integral by less than $\frac{1}{200}$. This question tested whether students could correctly use and justify the error bound for an alternating series whose terms are decreasing in absolute value to zero.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student has the first four nonzero terms and the correct general term so the first 3 points were earned. In part (b) none of the student's work earned any points. In part (c) the student makes an error in antidifferentiating the last term so only the first point for two terms was earned. The estimation is correct and earned the third point. In part (d) the student uses the third term as an error bound and successfully calculates $\frac{1}{320}$, and thus the first point was earned. The second point was not earned since the student explains that this is an alternating series but does not observe that the individual terms decrease in absolute value to 0.

Sample: 6C
Score: 4

The student earned 4 points: 1 point in part (a), no points in part (b), 3 points in part (c), and no points in part (d). In part (a) the student earned the first point for $1 - x^2$. The other terms are incorrect, and there is no general term so no other points were earned. In part (b), since the student's limit does not exist, the student was not eligible for the point. In part (c) the student correctly antidifferentiates the polynomial from part (a) so earned the first 2 points. The student's correct estimation earned the third point. In part (d) none of the student's work earned any points.